International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

Systems of Covariance in Relativistic Quantum Mechanics

S. Twareque Ali[1]

Received July 4, 1997

Developing some earlier work for spin-zero systems found in the literature, we use some recently obtained generalized systems of covariance for the Poincaré group to suggest a method for defining covariant localization operators on phase space for massive relativistic particles with arbitrary integral or half-integral spins. These operators lead to operationally defined position operators on spacelike hyperplanes, which turn out to be the Newton–Wigner operators, and, as in the earlier results on spin-zero systems, admit a consistent probability interpretation with conserved currents.

1. PRELIMINARIES ON SYSTEMS OF COVARIANCE

Systems of covariance usually arise in quantum mechanics as sets of localization operators on some parameter space, obeying specific transformation rules under the action of a kinematical symmetry group of the quantum system being considered (Ali, 1985). The mathematical precursor of this concept is that of a system of imprimitivity which arises in the theory of induced representations of groups, as worked out by Mackey (1968). Let G be a locally compact group, H a closed subgroup of G, and $X = G/H$ the associated left coset space. Let $g \mapsto U(g)$ be a (strongly continuous) unitary representation of G on the (separable, complex) Hilbert space $\mathfrak{h}$. Suppose that there is defined, on the Borel sets $\mathcal{B}(X)$ of X, a positive operator-valued (POV) measure a, in other words, a mapping, a: $\mathcal{B}(X) \to \mathcal{L}(\mathfrak{h})^+$, with the properties

$$a(\emptyset) = 0, \qquad a(X) = I = \text{identity operator on } \mathfrak{h}$$
$$a\left(\bigcup_{i\in J} \Delta_i\right) = \sum_{i\in J} a(\Delta_i) \tag{1.1}$$

[1] Department of Mathematics and Statistics, Concordia University, Montréal, Québec, Canada H4B 1R6.

where J is a discrete index set and $\Delta_i \cap \Delta_j = \emptyset$, for $i \neq j$, the convergence of the above sum being in the weak sense. For $g \in G$ and $\Delta \in \mathcal{B}(X)$, let $g\Delta$ be the translated set (under the natural action $x \mapsto gx$ of G on X). Then the pair $\{U, a\}$ is called a *system of covariance* if

$$U(g)a(\Delta)(g)^* = a(g\Delta), \qquad g \in G, \qquad \Delta \in \mathcal{B}(X) \tag{1.2}$$

In this case it can be proved (Neumann, 1972; Scutaru, 1977) that U is unitarily equivalent to a subrepresentation of an induced representation. In the more restrictive situation where a is a projection-valued (PV) measure, i.e., $a(\Delta) = P(\Delta)$, where $P(\Delta) = P(\Delta)^2 = P(\Delta)^*$, for all $\Delta \in \mathcal{B}(X)$, the pair $\{U, P\}$ is called a *system of imprimitivity* (Mackey, 1968) and in this case U is itself equivalent to an induced representation.

The operators $a(\Delta)$, defining a system of covariance, are usually localization operators for quantum systems moving in the parameter space X. Thus, if $\phi \in \mathfrak{h}$ is a state vector, then $p_\phi(\Delta) = \langle \phi | a(\Delta)\phi \rangle$ represents the probability of finding the quantum system, in the state ϕ, localized in the volume Δ of the parameter space X. In practice X could, for example, be the position, momentum, or phase space of the system. The covariance condition (1.2) is then just the transformation law of these probabilities under the action of the symmetry group G.

A rich source for building systems of covariance is provided by families of *coherent states* (Ali *et al.*, 1995). In fact, using families of coherent states, one can introduce a somewhat more general notion of a system of covariance than envisaged in (1.2). To see how this is done, assume that U is an *irreducible* unitary representation of G and that the coset space X carries the *invariant* measure $d\nu$ (under the natural action of G on X). Let σ: $X \to G$ be a (global) Borel section, i.e., $\sigma(x) \in G$ ($\forall x \in X$) and $\pi(\sigma(x)) = x$, where π: $G \to X$ is the canonical projection. Let η^i, $i = 1, 2, \ldots, n$, be a finite set of linearly independent vectors in $\mathfrak{h}$. The set of vectors

$$\mathfrak{S}_\sigma = \{\eta^i_{\sigma(x)} = U(\sigma(x))\eta^i | i = 1, 2, \ldots, n, x \in X\} \tag{1.3}$$

is called a family of coherent states if its linear span is dense in $\mathfrak{h}$. If, in addition,

$$\sum_{i=1}^{n} \int_X |\eta^i_{\sigma(x)}\rangle\langle\eta^i_{\sigma(x)}| \, d\nu(x) = A_\sigma \tag{1.4}$$

as a weak integral, where A_σ and A_σ^{-1} are both bounded operators, the coherent states $\mathfrak{S}_\sigma$ are said to form a *frame*. Furthermore, we say that the frame is *tight* if $A_\sigma = I$. For tight frames we can define the following generalized notion of a system of covariance: Let σ_g: $X \to G$ be the covariantly transformed section

$$\sigma_g(x) = g\sigma(g^{-1}x) = \sigma(x)h(g, g^{-1}x), \qquad g \in G, \qquad x \in X \tag{1.5}$$

where $h: G \times X \to H$ is the cocycle, $h(g, x) = \sigma(gx)^{-1} g\sigma(x)$. Then $\sigma_e = \sigma$ (e being the identity element of G); for each $g \in G$ and $\Delta \in \mathcal{B}(x)$, define the positive operator

$$a_{\sigma_g}(\Delta) = \sum_{i=1}^{n} \int_{\Delta} |\eta^i_{\sigma_g(x)}\rangle\langle\eta^i_{\sigma_g(x)}| \, d\nu(x) \tag{1.6}$$

It is easily verified that for each $g \in G$, the family of operators $a_{\sigma_g}(\Delta)$, $\Delta \in \mathcal{B}(X)$, defines a POV measure. Furthermore,

$$U(g)a_{\sigma}(\Delta)U(g)^* = a_{\sigma_g}(g\Delta) \tag{1.7}$$

a relation which now appears as a generalization of (1.2). If $K \subset G$ is a subgroup which stabilizes σ, i.e., $(\forall k \in K)\ \sigma_k = \sigma$, then restricted to K the pair $\{U, a\}$ is a system of covariance in the sense of (1.2).

In order to obtain a satisfactory theory of localization for relativistic particles, it has been realized for some time (see, for example, Ali, 1985) that it is necessary to work with sets of localization operators which satisfy the more general condition (1.7) rather than (1.2). We now indicate how it is possible to achieve this, using systems of covariance arising from the Poincaré group.

2. SYSTEMS OF COVARIANCE FOR THE POINCARÉ GROUP

The Poincaré group is the semi-direct product $\mathcal{P}^{\uparrow}_{+}(1, 3) = T^4 \oslash SL(2, \mathbb{C})$, where $T^4 \simeq \mathbb{R}_{1,3}$ is the group of space-time translations. Elements in $\mathcal{P}^{\uparrow}_{+}(1, 3)$ will be denoted by (a, A), with $a = (a_0, \mathbf{a}) \in \mathbb{R}_{1,3}$, $A \in SL(2, \mathbb{C})$. The element of the proper orthochronous Lorentz group corresponding to A will be denoted by Λ. Let $\mathcal{V}^+_m$ be the forward mass hyperboloid, $\mathcal{V}^+_m = \{k = (k_0, \mathbf{k}) \in \mathbb{R}^4 | k^2 = k_0^2 - \mathbf{k}^2 = m^2, k_0 > 0\}$. (We take $\hbar = c = 1$.) The unitary irreducible representations U^s_m of $\mathcal{P}^{\uparrow}_{+}(1, 3)$ which will concern us here are the ones which describe relativistic particles of mass $m > 0$ and spin $s = 0, 1/2, 1, 3/2, \ldots$. These representations are carried by the Hilbert spaces $\mathfrak{h}^s_m = \mathbb{C}^{2s+1} \otimes \mathcal{L}^2(\mathcal{V}^+_m, d\mathbf{k}/k_0)$ and are given by the unitary operators

$$(U^s_m(a, A)\phi)(k) = \exp[ik \cdot a]\, \mathcal{D}^s(h(k)Ah(\Lambda^{-1}k))\phi(\Lambda^{-1}k)$$

$$k \cdot a = k_0 a_0 - \mathbf{k} \cdot \mathbf{a} \tag{2.1}$$

($\phi \in \mathfrak{h}^s_m$) where $h(k)$ is the $SL(2, \mathbb{C})$ element corresponding to the Lorentz boost matrix for the 4-velocity k/m, and $\mathcal{D}^s$ is the usual $(2s + 1)$-dimensional spinor representation of $SU(2)$. We shall refer to the representation U^s_m also as the momentum space representation for a particle of mass $m > 0$ and spin s.

We shall identify the physical phase space of the system with the coset space

$$\Gamma = \mathscr{P}_+^\uparrow(1, 3)/T \otimes SU(2) \tag{2.2}$$

where T is the time translation subgroup. It can be shown that Γ admits a global parametrization $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^6$ in which $\mathbf{q}$ has the interpretation of a position and $\mathbf{p}$ of a momentum variable, and in terms of this parametrization the Lebesgue measure $d\mathbf{q}\ d\mathbf{p}$ is the invariant measure (Ali, 1979) for Γ. In order to build coherent states for the representations U_m^s, it is useful to consider a class of sections σ: $\Gamma \to \mathscr{P}_+^\uparrow(1, 3)$, called affine sections. To describe these, we first define the simplest member of the class, the *Galilean section* σ_0:

$$\sigma_0(\mathbf{q}, \mathbf{p}) = ((0, \mathbf{q}), h(p)), \qquad p = (\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p}) \tag{2.3}$$

As $(\mathbf{q}, \mathbf{p})$ varies through $\mathbb{R}^6$, the range of σ_0 in $\mathscr{P}_+^1(1,3)$ can be identified with $\mathbb{R}^3 \times \mathscr{V}_m^+$.

Any other affine section σ is then defined as

$$\sigma(\mathbf{q}, \mathbf{p}) = \sigma_0(\mathbf{q}, \mathbf{p})((f(\mathbf{q}, \mathbf{p}), \mathbf{0}), \mathbb{I}_2), \qquad \mathbb{I}_2 = 2 \times 2 \text{ unit matrix} \tag{2.4}$$

where f is a smooth affine function:

$$f(\mathbf{q}, \mathbf{p}) = \mathbf{q} \cdot \vartheta(\mathbf{p}) + \varphi(\mathbf{p}) \tag{2.5}$$

ϑ: $\mathbb{R}^3 \to \mathbb{R}^3$ and φ: $\mathbb{R}^3 \to \mathbb{R}$ are smooth functions, of which φ is arbitrary and ϑ is restricted by

$$\|\beta(\mathbf{p})\| < 1, \qquad \text{where} \quad \beta(\mathbf{p}) = \frac{p_0 \vartheta(\mathbf{p})}{m + \mathbf{p} \bullet \vartheta(\mathbf{p})} \tag{2.6}$$

$(\forall \mathbf{p} \in \mathbb{R}^3)$. It is then easily verified that the range of σ in $\mathscr{P}_+^\uparrow(1, 3)$ can be identified with $\cup_{p \in \mathscr{V}_m^+} \Sigma_{n(p)}$, where $\Sigma_{n(p)}$ is the spacelike hyperplane in Minkowski space consisting of all points $\hat{q} \in \mathbb{R}_{1,3}$ such that

$$n(p) \cdot \hat{q} = [n(p)]_0 \hat{q}_0 - \underline{n(p)} \cdot \underline{\hat{q}} = \frac{1}{m} [n(p) \cdot p] \varphi(\mathbf{p}) \tag{2.7}$$

The underline denotes the spatial part of a 4-vector. It can now be shown (Ali *et al.*, 1996) that if σ is an affine section, characterized by the functions β and φ, then the section $\sigma_{(a, \Lambda)}$, which is the transform of σ according to (1.5), is characterized by the functions

$$\beta'(\mathbf{p}) = \frac{\underline{\Lambda n(p)}}{[\Lambda n(p)]_0}, \qquad \varphi'(\mathbf{p}) = \frac{\underline{n(p) \cdot [m(\Lambda^{-1} a) + \varphi(\mathbf{p}) p]}}{[\Lambda \mathrm{n}(p)] \cdot p} \tag{2.8}$$

Consider next the set of vectors in the representation space $\mathbb{C}^{2s+1} \otimes L^2(\mathcal{V}_m^+, d\mathbf{k}/k_0)$:

$$\eta^i = e_i \otimes \eta, \qquad i = 1, 2, \ldots, 2s + 1 \tag{2.9}$$

where $\{e_i\}_{i=1}^{2s+1}$ is the canonical basis of $\mathbb{C}^{2s+1}$ and $\eta: \mathcal{V}_m^+ \to \mathbb{C}$ is a function satisfying the conditions

$$\int_{\mathbb{R}^3} |\eta(k)|^2 \, d\mathbf{k} < \infty, \qquad |\eta(Rk)|^2 = |\eta(k)|^2, \qquad R \in SO(3) \tag{2.10}$$

For an arbitrary affine section σ, define the coherent states

$$\eta_\sigma^i(\mathbf{q}, \mathbf{p}) = U_m^s(\sigma(\mathbf{q}, \mathbf{p}))\eta^i, \qquad i = 1, 2, \ldots, 2s + 1, \quad (\mathbf{q}, \mathbf{p}) \in \Gamma \tag{2.11}$$

A general result, proved in Ali *et al.* (1996; see also Prugovečki, 1980), then shows that these coherent states form a frame. In particular, there exist sections (e.g., the Galilei section σ_0 and its translates) for which the frame is tight:

$$\sum_{i=1}^{2s+1} \int_1 |\eta_{\sigma(\mathbf{q},\mathbf{p})}^i\rangle\langle\eta_{\sigma(\mathbf{q},\mathbf{p})}^i| \, d\mathbf{q} \, d\mathbf{p} = I \tag{2.12}$$

Furthermore, if σ is a section which generates a tight frame, then so also do all the transformed sections $\sigma_{(a,A)}$, $(a, A) \in \mathcal{P}_+^\uparrow(1, 3)$. The operators

$$a_{\sigma_{(a,A)}}(\Delta) = \sum_{i=1}^{2s+1} \int_\Delta |\eta_{\sigma(\mathbf{q},\mathbf{p})}^i\rangle\langle\eta_{\sigma(\mathbf{q},\mathbf{p})}^i| \, d\mathbf{q} \, d\mathbf{p} \tag{2.13}$$

form a generalized system of covariance:

$$U_m^s(a, A)a_\sigma(\Delta)U_m^s(a, A)^* = a_{\sigma_{(a,\Lambda)}}((a, A)\Delta) \tag{2.14}$$

Suppose now that we start with the Galilean section $\sigma = \sigma_0$. This section is characterized by the functions $\beta(\mathbf{p}) = \mathbf{0}$ and $\varphi(\mathbf{p}) = 0$, for all $\mathbf{p} \in \mathbb{R}^3$, and, as we saw before, its range in $\mathcal{P}_+^\uparrow(1, 3)$ can be identified with $\mathbb{R}^3 \times \mathcal{V}_m^+$. According to (2.8), the translated section $\sigma_{(n,A)}$ is characterized by

$$\beta(\mathbf{p}) = \frac{\underline{\Lambda(1, \mathbf{0})}}{[\Lambda(1, \mathbf{0})]_\mathbf{0}} = \beta, \qquad \varphi(\mathbf{p}) = \frac{m[\Lambda^{-1}a]_0}{[\Lambda(1, \mathbf{0})] \cdot \mathbf{p}} \tag{2.15}$$

Thus, the range of $\sigma_{(a,A)}$ in $\mathcal{P}_+^\uparrow(1, 3)$ can be identified with $\cup_{p \in \mathcal{V}_m^+} \Sigma_{n(p)}$, where $\Sigma_{n(p)}$ is the tilted hyperplane consisting of all $\hat{q} \in \mathbb{R}_{1,3}$ such that $n \cdot \hat{q} = [(n \cdot p)/m]\varphi(\mathbf{p})$, $n = \Lambda(1, \mathbf{0})$. We also note from here that if we restrict ourselves to the Euclidean subgroup $\mathscr{E} \subset \mathcal{P}_+^\uparrow(1, 3)$, consisting of all group elements of the type $((0, \mathbf{a}), \rho)$, were $\mathbf{a} \in \mathbb{R}^3$ and $\rho \in SU(2)$, then σ_0 is stable under its action and consequently, for this subgroup (2.14) reduces to a system of covariance in the sense of (1.2). Recall that we identify $\Gamma = \mathcal{P}_+^{\uparrow}(1,3)/T \otimes SU(2)$ with the phase space of the relativistic system, and therefore the system of covariance (2.14) is based on phase space.

3. RELATIVISTIC LOCALIZATION ON PHASE SPACE

We demonstrate in this section how the generalized system of covariance introduced in (2.14) enables us to discuss the localizability of massive relativistic particles with spin in a manner which is operationally consistent with the existence of a conserved probability current and a position operator. The traditional way of understanding relativistic localization follows the original suggestion of Newton and Wigner (1949) as later elaborated and interpreted group theoretically by Wightman (1962). This interpretation is linked to the existence of a system of imprimitivity for the Euclidean group $\mathscr{E}$, based upon the configuration space $\mathbb{R}^3$. In order to understand this in our present context, let us consider again the representation U_m^s in (2.1) and this time look at its position space realization. Let $\mathscr{F}$: $\mathbb{C}^{2s+1} \otimes \mathscr{L}^2(\mathscr{V}_m^+, d\mathbf{k}/k_0) \to \mathbb{C}^{2s+1} \otimes L^2(\mathbb{R}^3, d\mathbf{x})$ be the "weighted Fourier transform":

$$(\mathscr{F}\phi)(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \exp[i\mathbf{k} \cdot \mathbf{x}]\, \phi(\mathbf{k}) \frac{d\mathbf{k}}{(\mathbf{k}^2 + m^2)^{1/2}} \tag{3.1}$$

This map is unitary; let $U_{\text{pos}}(a, A) = \mathscr{F} U_m^s(a, A) \mathscr{F}^{-1}$ be the corresponding unitarily equivalent representation of $\mathcal{P}_+^{\uparrow}(1, 3)$ on $\mathbb{C}^{2s+1} \otimes L^2(\mathbb{R}^3, d\mathbf{x})$. Elements $\psi \in \mathbb{C}^{2s+1} \otimes L^2(\mathbb{R}^3, d\mathbf{x})$ represent position-space wave functions. For each Borel set $E \subset \mathbb{R}^3$, consider the projection operator $P(E)$ on $\mathbb{C}^{2s+1} \otimes L^2(\mathbb{R}^3, d\mathbf{x})$:

$$(P(E)\psi)(\mathbf{x}) = \chi_E(\mathbf{x})\psi(\mathbf{x}), \qquad \psi \in \mathbb{C}^{2s+1} \otimes L^2(\mathbb{R}^3, d\mathbf{x}) \tag{3.2}$$

χ_E is the characteristic function of the set E. These operators constitute a PV-measure. Moreover, since for $((0, \mathbf{a}), \rho) \in \mathscr{E}$,

$$(U_{\text{pos}}((0, \mathbf{a}), \rho)\psi)(\mathbf{x}) = \mathscr{D}^s(\rho)\psi(R(\rho^{-1})(\mathbf{x} - \mathbf{a})), \qquad \psi \in \mathbb{C}^{2s+1} \otimes L^2(\mathbb{R}^3, d\mathbf{x}) \tag{3.3}$$

where $R(\rho)$ denotes the rotation matrix in $SO(3)$ corresponding to $\rho \in SU(2)$, we easily verify the relation

$$U_{\text{pos}}((0, \mathbf{a}), \rho)P(E)U_{\text{pos}}((0, \mathbf{a}), \rho)^* = P(((0, \mathbf{a}), \rho)E) \tag{3.4}$$

Thus, restricted to $\mathscr{E}$, the representation U_{pos} and the operators $P(E)$ constitute a system of imprimitivity. Moreover, defining the three *position operators* $\hat{Q}_i$ and their inverse transformed versions Q_i,

$$\hat{Q}_i = \int_{\mathbb{R}^3} x_i \, dP(\mathbf{x}), \qquad Q_i = \mathscr{F}^{-1}\hat{Q}_i\mathscr{F}, \qquad i = 1, 2, 3 \tag{3.5}$$

it is not hard to see that the Q_i are exactly the Newton–Wigner (1949) position operators for a relativistic particle of mass $m > 0$ and spin $s = 0, 1/2, 1, 3/2, \ldots$. It is the existence of these position operators along with the system of imprimitivity (3.4) for the Euclidean subgroup $\mathscr{E}$ of $\mathscr{P}^1_+(1, 3)$ which forms the basis of the Newton–Wigner–Wightman theory of localizability for such a particle.

Although the above scheme is satisfactory insofar as the existence of the position operators (3.5) is concerned, there is a difficulty with a probability interpretation of the operators $P(E)$. For example, the quantity $\langle \psi | P(E)\psi \rangle$ cannot be interpreted as the probability of finding the system (in the state ψ) localized in the volume E of position space. Furthermore, if $\psi(\mathbf{x}, t)$ is the time-translated wave function, then $\|\psi(\mathbf{x}, t)\|^2$ is not the time component of any conserved probability current.

Let us see how the operators $a_{\sigma_{(a,A)}}(\Delta)$ appearing in the generalized system of covariance (2.14) enable us to overcome these difficulties. Let us agree to call the operators $a_{\sigma_0}(\Delta)$ [defined for the Galilean section, see (2.13)] the operators of localization on the relativistic phase space $\mathbb{R}^3 \times \mathscr{V}^+_m$. In that case, the operators $a_{\sigma_{(0,A)}}(\Delta)$ represent localization operators on the Lorentz-transformed phase space $\Sigma_n \times \mathscr{V}^+_m$, where Σ_n is the tilted spacelike hyperplane in Minkowski space, the points $\hat{q}$ of which satisfy $\hat{q} \cdot n = 0$, where $n = \Lambda(1, \mathbf{0})$. We shall interpret $\langle \phi | a_{\sigma_{(0,A)}}(\Delta)\phi \rangle$ as the probability of finding the system (in the state ϕ) localized in the volume $\hat{\Delta} \subset \Sigma_n \times \mathscr{V}^+_m$, where

$$\begin{aligned} \hat{\Delta} = \{(\hat{q}, p) \in \Sigma_n \times \mathscr{V}^+_m | \hat{q} = \Lambda^{-1}(0, \mathbf{q}), \\ p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p}), (\mathbf{q}, \mathbf{p}) \in \Delta\} \end{aligned} \tag{3.6}$$

Next, a straightforward computation shows that the Newton–Wigner operators Q_i in (3.5) can also be recovered by integrating over the phase-space position variables with respect to the POV-measure a_{σ_0}, i.e.,

$$Q_i = \int_\Gamma q_i \, da_{\sigma_0}(\mathbf{q}, \mathbf{p}), \qquad i = 1, 2, 3 \tag{3.7}$$

More interestingly, if $(\mathbf{q}', \mathbf{p}') = (0, \Lambda^{-1})(\mathbf{q}, \mathbf{p})$, then

$$U^m_s(0, A)Q_iU^m_s(0, A)^* = \int_\Gamma q_i' \, da_{\sigma_{(0,\Lambda)}}(\mathbf{q}, \mathbf{p}), \qquad i = 1, 2, 3 \tag{3.8}$$

In other words, the transformed position operator is obtained simply by integrating with respect to the variables of the tilted hyperplane.

Finally, note that the map W: $\mathbb{C}^{2s+1} \otimes L^2(\mathcal{V}_m^+, d\mathbf{k}/k_0) \to \mathbb{C}^{2s+1} \otimes L^2(\Gamma, d\mathbf{q}\, d\mathbf{p})$:

$$(W\phi)^i(\mathbf{q}, \mathbf{p}) = \langle \eta^i_{\sigma_0(\mathbf{q},\mathbf{p})} | \phi \rangle := \Psi^i(\mathbf{q}, \mathbf{p}), \qquad i = 1, 2, \ldots, 2s + 1 \quad (3.9)$$

is an isometric embedding. We call $\boldsymbol{\Psi}(\mathbf{q}, \mathbf{p})$, with components $\Psi^i(\mathbf{q}, \mathbf{p})$, a phase-space wave function and $\|\boldsymbol{\Psi}(\mathbf{q}, \mathbf{p})\|^2$ a phase-space probability density. If H_0 denotes the free Hamiltonian on $\mathbb{C}^{2s+1} \otimes L^2(\mathcal{V}_m^+, d\mathbf{k}/k_0)$, $(H_0\phi)(\mathbf{k}) = k_0\phi$, then the function $\boldsymbol{\Psi}(\mathbf{q}, \mathbf{p}, t)$, with components,

$$\Psi^i(\mathbf{q}, \mathbf{p}, t) = \langle \eta^i_{\sigma_0(\mathbf{q},\mathbf{p})} | e^{-iH_0} \phi \rangle, \qquad i = 1, 2, \ldots, 2s + 1 \quad (3.10)$$

is a time-translated phase-space wave function. At this point, let us assume that the function η appearing in the definition of η^1 in (2.9) is real-valued, and define a current (Prugovecki, 1978a, b)

$$j_\mu(q) = \int_{V_m^+} \frac{p_\mu}{m} \|\boldsymbol{\Psi}(\mathbf{q}, \mathbf{p}, t)\|^2 \frac{d\mathbf{p}}{\mathbf{p}_0},$$

$$q = (q_0, \mathbf{q}) = (t, \mathbf{q}),$$

$$\mu = 0, 1, 2, 3 \quad (3.11)$$

It can then be shown (Ali *et al.*, 1988; Prugovecki, 1978a, b) that j_μ transforms as a 4-vector under Lorentz transformations and

$$\partial^\mu j_\mu(q) = 0 \quad (3.12)$$

i.e., j_μ represents a conserved current. This shows that the generalized relativistic system of covariance (2.14) leads to an entirely consistent notion of localization on phase space. Moreover, this interpretation is supported by the existence of operationally defined position operators and conserved currents, which transform properly under the Lorentz group. This latter property of current conservation was first obtained in Prugovečki (1978a, b) for a spin-zero particle, where, indeed, the suggestion was first made that relativistic particles be localized on phase space rather than on position space alone.

In conclusion, it ought to be emphasized that admitting localization operators which are POV-measures, as opposed to PV-measures, was the crucial element in circumventing the shortcomings of the Newton–Wigner–Wightman scheme of localization for massive relativistic particles with spin.

REFERENCES

Ali, S. T. (1979). On some representations of the Poincaré group on phase space, *Journal of Mathematical Physics,* **20**, 1385–1391.

Ali, S. T. (1985). Stochastic localization, quantum mechanics on phase space and quantum space-time, *Rivista del Nuovo Cimento,* **8**, 1–128.

Ali, S. T., Brooke, J. A., Busch, P., Gagnon, R., and Schroeck, F. E. Jr., (1988). Current conservation as a geometric property of space-time, *Canadian Journal of Physics,* **66**, 238–244.

Ali, S. T., Antoine, J.-P., Gazeau, J.-P., and Mueller, U. A. (1995). Coherent states and their generalizations: A mathematical overview, *Review of Mathematical Physics,* **7**, 1013–1104.

Ali, S. T., Gazeau, J.-P., and Karim, M. R. (1996). Frames, the β-duality in Minkowski space and spin coherent states, *Journal of Physics, A,* **29**, 5529–5549.

Mackey, G. W. (1968). *Induced Representations of Groups and Quantum Mechanics*, Benjamin, New York.

Neumann, H. (1972). Transformation properties of observables, *Helvetica Physica Acta,* **45**, 811–819.

Newton, T. D., and Wigner, E. P. (1949). Localized states for elementary systems, *Reviews of Modern Physics,* **21**, 400–406.

Prugovečki, E. (1978a). Consistent formulation of relativistic dynamics for massive spin-zero particles in external fields, *Physical Review D,* **18**, 3655–3673.

Prugovečki, E. (1978b). Relativistic quantum kinematics on stochastic phase space for massive particles, *Journal of Mathematical Physics,* **19**, 2261–2270.

Prugovečki, E. (1980). Dirac dynamics on stochastic phase space for spin-1/2 particles, *Reports on Mathematical Physics,* **17**, 401–417.

Scutaru, H. (1977). Coherent states and induced representations, *Letters on Mathematical Physics,* **2**, 101–107.

Wightman, A. S. (1962). On the localizability of quantum mechanical systems, *Reviews of Modern Physics,* **34**, 845–872.